Imperial College London

Lecture 6

Linearity, Superposition & Thevenin Equivalent Circuits

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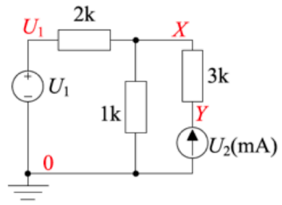
In this lecture, we will develop two important concepts relating to resistor circuits: 1) The concept of LINEARITY; 2) the concept of SUPERPOSITION.

I will approach these from a general point of view and then show you how these two concept may be used in understanding and working with electronic circuits, or indeed, many engineering systems.

Linearity Theorem

- Suppose we use variables instead of fixed values for all of the *independent* voltage and current sources. We can then use nodal analysis to find all node voltages in terms of the source values.
  1. Label all the nodes
  2. KCL equations: $\frac{X-U_1}{2}+\frac{X}{1}+\frac{X-Y}{3}=0$
     $\frac{Y-X}{3}+(-U_2)=0$
  3. Solve for the node voltages
     $X=\frac{1}{3}U_1+\frac{2}{3}U_2, \quad Y=\frac{1}{3}U_1+\frac{11}{3}U_2$



- Steps (2) and (3) never involve multiplying two source values together, so:

> **Linearity Theorem**: For any circuit containing resistors and independent voltage and current sources, every node voltage and branch current is a linear function of the source values and has the form $\sum a_i U_i$ where the U_i are the source values and the a_i are suitably dimensioned constants.

- Also true for a circuit containing dependent sources providing their values are sums of multiples of other voltages and/or currents in the circuit.

Let us take one of the circuits we have seen before, and label the voltage and current sources with variables U1 and U2.

Apply KCL at node X and Y. You can now solve for unknowns X and Y in terms of U1 and U2.

Since U1 and U2 are energy **sources**, they are the "**causes**" of voltages at X and Y. These voltages are the "**effect**".

What the equation in step 3 shows is that X and Y are always **WEIGHTED SUMS** of the sources. The equations shown here hold LINEAR relationship between the causes (U1, U2) and the effects (X, Y).

We say that this circuit is a **LINEAR** circuit because all node voltages and branch currents are linearly related to the the sources.

Implications of Linearity

- A linear circuit can therefore be described as:

  Effect = **Linear Function F** (Causes) or

  $V_{effect} = a_1$ x Cause_1 + a_2 x Cause_2 + ……. (a_1, a_2 ... are constants)

- There are TWO important properties in a linear circuits:
  1. **Proportionality** – If you multiply a **cause** by a factor M, the **effect** is also multiplied by the same factor M.
  2. **Superposition** – You can find the effects produce by two causes SEPARATELY, and COMBINE (i.e. add) them together to find the effect of both causes. In other words:

     If Effect_1 = F (Cause_1)

     Effect_2 = F (Cause_2), where F is a **linear function** (i.e. linear circuit)

     then Total Effect = F (Cause_1 + Cause_2) = Effect_1 + Effect_2

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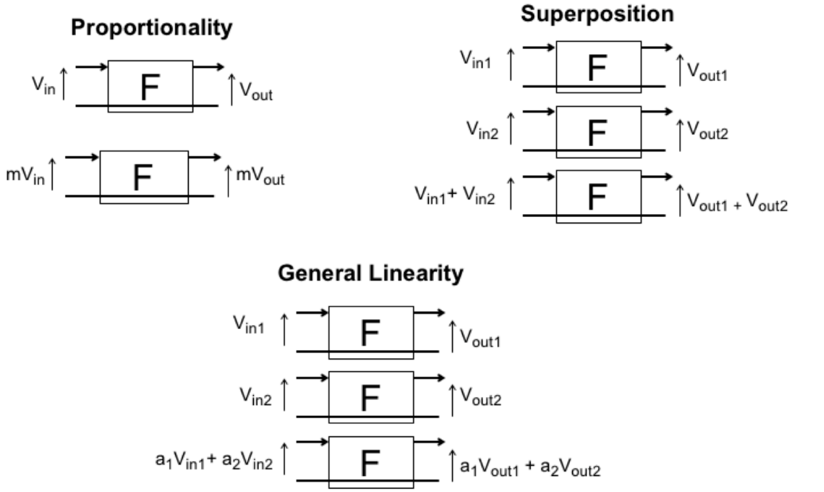
So, linearity is characterised by a linear function which is a weighted sum of variables (causes) where the weights are constants.

Linearity in a system results in two important properties. They are: 1) **Proportionality**, and 2) **Superposition**.

Proportionality means that if you double to cause, the effect also doubles.

Superposition means a the combined effect due to two (or more) causes is the sum of the effects for each cause taken separately.

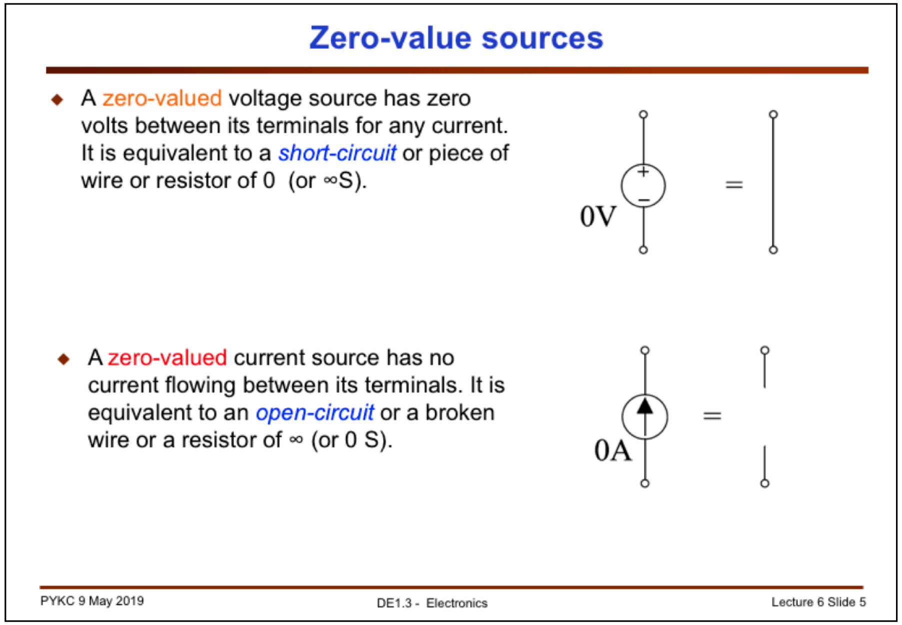
Implications of Linearity



This is best shown diagrammatically. Although what is shown here are input and output voltages to a system, the same holds true for any node voltages or currents inside a circuit, and for any fixed or dependent sources (provided that the defining equation of the dependent sources also has a linear relationship).

Combining the **proportionality** and the **superposition** principles together give us the general **linearity property**, which is modelled by the weighted sum equation:

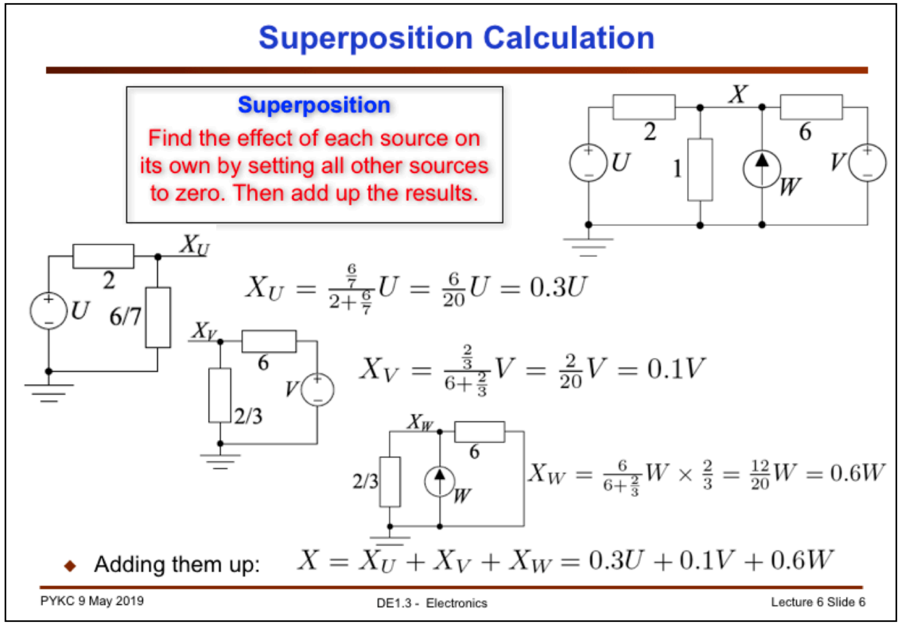
$$X = \Sigma\, a_i U_i$$



Let me further introduce the idea of zero-value sources. A voltage source could have a zero value. In which case, it is equivalent to a short circuit or a wire. If you have a voltage source with a fixed value V_S, and if you wish to "eliminate" its impact (cause) on a circuit, you can regard that V_S=0, or replacing the voltage source with a wire.

If a current source gives 0A, it is equivalent to an OPEN CIRCUIT (with infinite resistance). If you have a current source with a fixed value I_S, and if you wish to "eliminate" its impact (cause) on a circuit, you can regard that I_S=0, or removing the current source from the circuit all together.

Why are these two concept useful? You will see later.



Let us apply the principle of superposition to evaluation X in the circuit shown here.

Step 1: Consider only source U and zero both V and W. Zero V means replacing it with a wire. Zero W mean removing W altogether. This results in resistor 1 || 6 = 6/7. Therefore X_U = 0.3U.

Step 2: Consider only source V and zero both U and W. We short U to ground and remove W. This results in resistor 2 || 1 = 2/3. Therefore X_V = 0.1V.

Step 3: Consider only source W and zero both U and V. We short both U and V. Therefore X_W = 0.6W.

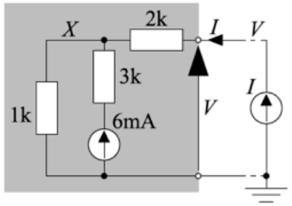
Using superposition X_{total} = 0.3U + 0.1V + 0.6W

Equivalent Networks

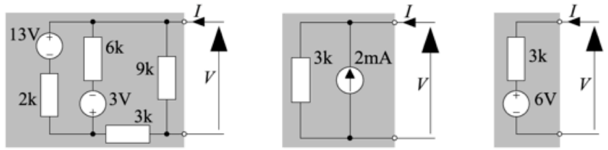
- From linearity theorem: $V = aI + b$.
- Use nodal analysis:

  KCL@X: $\frac{X}{1} - 6 + \frac{X-V}{2} = 0$

  KCL@V: $\frac{V-X}{2} - I = 0$



- Eliminating X gives: $V = 3I + 6$.
- There are infinitely many networks with the same values of a and b:



- These four shaded networks are *equivalent* because the relationship between V and I is *exactly* the same in each case.
- The last one is particularly simple and is called the *Thévenin* equivalent network.

Let us consider the circuit shown here. Imagine that circuit in gray is a "**black box**", and we are interested in the relationship between the voltage V at the output of the black box, and the current I flowing into the black box. We can model this situation with the unknown current source I at the output of the black box.

From slide 8 of this lecture, we know that since there is only one unknown source I, By applying linearity theorem, we get V = aI + b. Perform KCL at X and V, we get V = 3I + 6.

Interestingly, we can replace the gray circuit (the black box) with many different circuits that have exactly the same V-I relationship.

Shown below are three alternatives which are equivalent to the circuit on the top. They are equivalent because they all have the same V vs I equation.

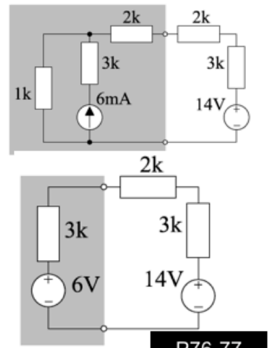
The right-most circuit, which could replace all the other equivalent circuit is simple and most important, and it is known as the **Thévenin equivalent circuit**.

Thévenin Equivalent Networks

Thévenin Theorem

Any two-terminal network consisting of resistors, fixed voltage/current sources and linear dependent sources is externally equivalent to a circuit consisting of a resistor in series with a fixed voltage source.

- We can replace the shaded part of the circuit with its Thévenin equivalent network.
- The voltages and currents in the unshaded part of the circuit will be identical in both circuits.
- The new components are called the *Thévenin equivalent resistance*, R_{Th}, and the *Thévenin equivalent voltage*, V_{Th}, of the original network.
- This is often a useful way to simplify a complicated circuit (provided that you do not want to know the voltages and currents inside the shaded part).



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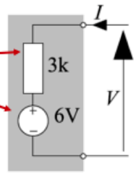
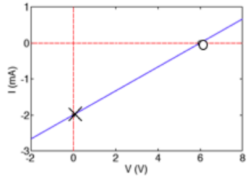
Thévenin Theorem states: consider a two terminal circuit (network) with only resistors, fixed sources and linear dependent sources. No matter how complex this circuit is, you can always replace it with a simple fixed voltage source in series with a resistor. So the upper shaded circuit can be replaced with the lower shaded circuit.

Thévenin Circuit Properties

- A Thévenin equivalent circuit has a straight line characteristic with the equation: $V = R_{Th} I + V_{Th}$

  $\Leftrightarrow I = \frac{1}{R_{Th}} V - \frac{V_{Th}}{R_{Th}}$





- Three important quantities are:
  - Open Circuit Voltage: If $I = 0$ then $V_{OC} = V_{Th}$. (X-intercept: o)
  - Short Circuit Current: If $V = 0$ then $I_{SC} = -\frac{V_{Th}}{R_{Th}}$ (Y-intercept: x)
  - Thévenin Resistance: The slope of the characteristic is. $\frac{dI}{dV} = \frac{1}{R_{Th}}$.
- If we know the value of any two of these three quantities, we can work out V_{Th} and R_{Th}.
- In any two-terminal circuit with the same characteristic, the three quantities will have the same values. So if we can determine two of them, we can work out the Thévenin equivalent.

Thévenin equivalent circuit always have a straight line characteristic. (Characteristic of a component or an electronic widget is the plot between I-V or V-I, or some other electrical parameters). In a Thévenin equivalent circuit,

$V = R_{TH} I + V_{TH}$

V_{TH} = Thévenin voltage, R_{TH} = Thévenin Resistance.

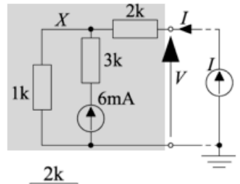
The Thévenin voltage can be found by open circuit (i.e. not connect anything) the shaded circuit. Then I = 0, and $V = V_{TH}$.

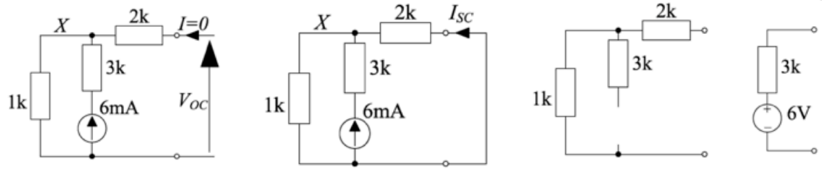
R_{TH}, is simply the 1/ gradient of the curve, or 1/ dI/dV.

Alternative, you can think of R_{TH} as the resistance of the shaded circuit when V_{TH} is zero.

Determining Thévenin Values

- We need any two of the following:
  - Open Circuit Voltage: $V_{OC} = V_{Th} = 6\,\text{V}$
  - Short Circuit Current: $I_{SC} = -\frac{V_{Th}}{R_{Th}} = -2\,\text{mA}$
  - Thévenin Resistance: $R_{Th} = 2\,\text{k} + 1\,\text{k} = 3\,\text{k}\Omega$





Thévenin Resistance:

- We set all the independent sources to zero (voltage sources → short circuit, current sources → open circuit). Then we find the equivalent resistance between the two terminals.
- The $3\,k$ resistor has no effect so $R_{Th} = 2\,k + 1\,k = 3\,k$.
- Any measurement gives the same result on the equivalent circuit.

Let us consider how to go from the complicated circuit on the upper right to the simple Thévenin equivalent circuit on the lower right.

Remember,

$V = R_{TH} * I + V_{TH}$

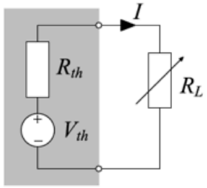
Step 1: open circuit the output (i.e. I = 0). Find V_{OC}.

Step 2: we short circuit the output (i.e. V = 0), and find the current I.

Step 3: we zero the source (i.e. the 6mA source become 0mA) and in this case, the current source is removed. Then compute the resistance between the two terminals.

Power Transfer

- Suppose we connect a variable resistor, RL, across a two-terminal network.
- From Thévenin's theorem, even a complicated network is equivalent to a voltage source and a resistor.



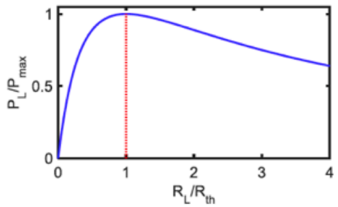
- We know $I = \frac{V_{Th}}{R_{Th}+R_L}$
  $\Rightarrow$ power in R_L is $P_L = I^2 R_L = \frac{V_{Th}^2 R_L}{(R_{th}+R_L)^2}$
- To find the R_L that maximizes P_L:

$$0 = \frac{dP_L}{dR_L} = \frac{(R_{Th}+R_L)^2 V_{Th}^2 - 2V_{Th}^2 R_L (R_{Th}+R_L)}{(R_{Th}+R_L)^4}$$

$$= \frac{V_{Th}^2 (R_{Th}+R_L) - 2V_{Th}^2 R_L}{(R_{Th}+R_L)^3}$$

$$\Rightarrow V_{Th}^2 \left((R_{Th}+R_L) - 2R_L\right) = 0$$

$$\Rightarrow R_L = R_{Th} \Rightarrow P_{(max)} = \frac{V_{Th}^2}{4R_{Th}}$$



- For fixed R_{Th}, the maximum power transfer is when $R_L = R_{Th}$ ("matched load").

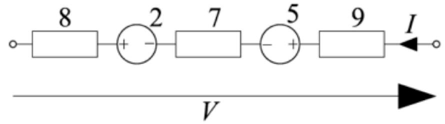
Series Rearrangement

- If we have any number of voltage sources and resistors in series we can calculate the total voltage across the chain as:

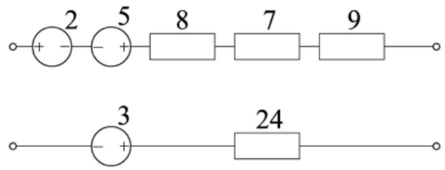
$$V = 8I - 2 + 7I + 5 + 9I = (-2+5) + (8+7+9)I$$

$$= 3 + 24I$$

- We can arbitrarily rearrange the order of the components without affecting V = 3 + 24I.



- If we move all the voltage sources together and all the resistors together we can merge them and then we get the Thévenin equivalent.



Let us consider a source with some internal resistance modelled by the Thévenin equivalent circuit as shown here. What is value is the resistor load such that the power transferred from the source to the load resistor is MAXIMUM?

The calculation here shows that maximum power is transferred if the load RL matches that of the Thévenin resistance. R_{TH}. This is known as Thévenin Thévenin .

Here are some examples of how to re—arrange sources and resistances to simply circuits.

Summary

- Linearity Theorem: $X = \sum_i a_i U_i$ 'er all independent sources U_i
- Proportionality: multiplying all sources by k multiplies all voltages and currents by k and all powers by k^2.
- Superposition: sometimes simpler than nodal analysis, often more insight.
  - Zero-value voltage and current sources
  - Dependent sources - treat as independent and add dependency as an extra equation
- If all sources are fixed except for U_1 then all voltages and currents in the circuit have the form $aU_1 + b$.
- Power does not obey superposition.
- Thévenin Equivalent Circuits
  - How to determine V_{Th}, I_{NO} and R_{Th}
    - Method 1: Nodal analysis
    - Method 2: Find any two of $V_{OC} = V_{Th}$, I_{SC}
    - R_{Th} is the equivalent resistance with all sources set to zero
  - Ohm's law is satisfied: $V_{Th} = I_{NO}R_{Th}$
  - Load resistor for maximum power transfer = R_{Th}